Oscillating Airfoils: I. Wedges of Arbitrary Thickness in Supersonic and Hypersonic Flow

Ronald M. Barron*
University of Windsor, Windsor, Ontario, Canada

Equations are obtained, by a perturbation method, for straight and curved airfoils with attached shocks in supersonic and hypersonic flows and undergoing small harmonic oscillations. The mode shape of the oscillation is arbitrary and hence may be used to describe rigid body motions as well as elastic deformations of the body surface. The wedge problem is solved for small frequency and an exact expression for the pressure distribution is obtained as a function of the mode of oscillation. The analysis presented can be easily modified to treat the case of arbitrary two-dimensional body shapes by developing an unsteady Newtonian flow theory. These results are reported in a subsequent paper.

I. Introduction

PERFORMANCE and stability of hypersonic and supersonic flight vehicles are of major concern to those involved in research and construction of guided missiles and re-entry spacecraft. From a practical point of view one requires a theory that adequately predicts the pressure distribution and hence the lift and stability derivatives of oscillating bodies. Such unsteady problems were initially considered by Lighthill[1] using piston theory, and by Zartarian et al.[2] and Miles[3] using shock-expansion theory. These theories, however, neglected the effects of secondary waves generated by the unsteady body motion and reflected from the bow shock wave. More accurate theories, taking this effect into account, have been developed by McIntosh[4] and Appleton[5] for two-dimensional thin wedges undergoing small harmonic oscillations in hypersonic flow. In a series of reports and papers, Hui[6,7] has lifted the "thinness" restriction of the wedge, while requiring that the shock wave remain attached at the nose. He has considered hypersonic flow past wedges undergoing small harmonic oscillations of an arbitrary nature in Ref. 6, and developed a unified theory for both supersonic and hypersonic flow past rigid wedges and caret wings oscillating in pitch in Ref. 7. In practice, due mainly to design and manufacturing problems and high temperatures associated with hypersonic flow, hypersonic vehicles will have slight blunting at the nose, hence restricting the applicability of the aforementioned theories.

In this paper a method of analysis is developed which can be applied to, or readily modified so as to apply to, arbitrary airfoil shapes, sharp or blunted, provided the shock wave emanating from the nose remains attached. In particular, we present a convenient matrix formulation which is applicable to any airfoil shape and valid for arbitrary mode of oscillation. Hence, pitching and plunging motions and elastic deformations of a flexible airfoil surface may be considered. These equations are solved for the case of a wedge with arbitrary mode of oscillation. These results may be viewed as generalizations of results obtained by Carrier,[8] Van Dyke,[9] and Hui.[7] The method developed here has also been applied to airfoils supporting power-law shocks, and the results are reported in a subsequent paper.[10]

II. Mathematical Formulation of the Problem

Consider a two-dimensional symmetrical airfoil of chord $\bar{L}$ undergoing small harmonic oscillations of a general nature in a uniform stream of gas of velocity $\bar{U}_\infty$. Let the airfoil be situated so that its axis of symmetry, when in its mean position, is parallel to the freestream direction. Assume that the flow is inviscid, the effects of heat conduction are negligible and the gas is perfect, with constant adiabatic exponent γ.

A Cartesian coordinate system $(\bar{x},\bar{y})$ is chosen so that the origin coincides with the leading edge of the airfoil in its mean postion and the $\bar{x}$ axis lies along the direction of the freestream. If the airfoil is sharp-nosed, the resulting shock wave is attached at the nose, and the flow over the upper surface is independent of that over the lower surface. If the nose is blunt, the flows on the upper and lower surfaces are still independent, provided we consider the shock to remain attached. In the subsequent formulation we assume this to be the case.

The mathematical problem is to determine velocity, pressure, and density $\bar{u},\bar{v},\bar{p}$, and $\bar{\rho}$ throughout the region between the airfoil and the shock. The shock location is also unknown and must be determined as part of the solution. These quantities are governed by the familiar unsteady flow equations. The boundary conditions to be satisfied at the shock, defined by $\bar{y}=\bar{y}_s(\bar{x},\bar{t})$, are the usual Rankine-Hugoniot conditions at an unsteady oblique shock. At the airfoil surface, $\bar{y}=\bar{y}_b(\bar{x},\bar{t})$, the relative normal velocity of the fluid must vanish. Thus, at the surface,

$$\bar{v}(\bar{x},\bar{y},\bar{t}) = \frac{\partial \bar{y}_b}{\partial \bar{t}} + \bar{u}(\bar{x},\bar{y},\bar{t})\,\frac{\partial \bar{y}_b}{\partial \bar{x}} \tag{1}$$

The condition that the shock remains attached at the leading edge is:

$$\bar{y}_s(0,\bar{t}) = \bar{y}_b(0,\bar{t}) \tag{2}$$

and

$$\frac{\partial \bar{y}_s}{\partial \bar{t}} = \frac{\partial y_b}{\partial \bar{t}} \quad \text{at } \bar{x}=0 \tag{3}$$

A. Transformation of Governing Equations and Boundary Conditions

We descale the physical "barred" variables according to

$$x=\bar{x}/\bar{L}, \quad y=\bar{y}/\bar{L}, \quad t=\bar{t}\bar{U}_\infty/\bar{L}$$

$$u=\bar{u}/\bar{U}_\infty, \quad v=\bar{v}/\bar{U}_\infty, \quad \rho=\bar{\rho}/\bar{\rho}_\infty, \quad p=\bar{p}/(\bar{\rho}_\infty \bar{U}_\infty^2) \tag{4}$$

It is convenient for later work to define a new independent variable by

$$y^* = y - y_b(x,t) \tag{5}$$

Received Sept. 2, 1977; revision received Feb. 28, 1978.

Index categories: Supersonic and Hypersonic Flow; Nonsteady Aerodynamics.

*Assistant Professor, Dept. of Mathematics. Member AIAA.

to replace y. The equation of the airfoil surface is then transformed to $y^*=0$. Equation (5), in conjunction with Eq. (1), suggests defining a new dependent variable

$$v^*(x,y^*,t)=v(x,y^*,t)-\frac{\partial y_b}{\partial t}-u(x,y^*,t)\frac{\partial y_b}{\partial x} \tag{6}$$

to replace v, so that $v^*(x,y^*,t)=0$ at $y^*=0$.

With ρ,u,v^*,p considered as functions of x,y^*,t, the equations of motion are:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y^*}(\rho v^*)=0 \tag{7a}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v^*\frac{\partial u}{\partial y^*}+\frac{1}{\rho}\frac{\partial p}{\partial x}-\frac{1}{\rho}\frac{\partial y_b}{\partial x}\frac{\partial p}{\partial y^*}=0 \tag{7b}$$

$$\frac{\partial v^*}{\partial t}+\frac{\partial y_b}{\partial x}\frac{\partial u}{\partial t}+u\frac{\partial y_b}{\partial x}\frac{\partial u}{\partial x}+v^*\frac{\partial y_b}{\partial x}\frac{\partial u}{\partial y^*}+u\frac{\partial v^*}{\partial x}$$
$$+v^*\frac{\partial v^*}{\partial y^*}+2\frac{\partial^2 y_b}{\partial x\partial t}u+\frac{\partial^2 y_b}{\partial x^2}u^2+\frac{1}{\rho}\frac{\partial p}{\partial y^*}=-\frac{\partial^2 y_b}{\partial t^2} \tag{7c}$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v^*\frac{\partial}{\partial y^*}\right)\left(\frac{p}{\rho^\gamma}\right)=0 \tag{7d}$$

These equations must be solved subject to the condition at the surface

$$v^*(x,0,t)=0 \tag{8}$$

and the conditons at the shock. The shock conditions are algebraic equations which may be solved for the four quantities ρ,u,v^*,p at the shock

$$y^*=y_s^*(x,t)\equiv y_s(x,t)-y_b(x,t) \tag{8*}$$

to obtain

$$\rho(x,y^*,t)\Big|_{y^*=y_s^*}=\frac{\dfrac{\gamma+1}{\gamma-1}\left\{\dfrac{\partial y_s}{\partial x}+\dfrac{\partial y_s}{\partial t}\right\}^2}{\left\{\dfrac{\partial y_s}{\partial x}+\dfrac{\partial y_s}{\partial t}\right\}^2+\dfrac{2}{(\gamma-1)M_\infty^2}\left\{\left(\dfrac{\partial y_s}{\partial x}\right)^2+1\right\}} \tag{9a}$$

$$u(x,y^*,t)\Big|_{y^*=y_s^*}=1-\frac{\partial y_s}{\partial x}\frac{2}{(\gamma+1)}\left\{\frac{\dfrac{\partial y_s}{\partial x}+\dfrac{\partial y_s}{\partial t}}{\left(\dfrac{\partial y_s}{\partial x}\right)^2+1}-\frac{1}{M_\infty^2\left(\dfrac{\partial y_s}{\partial x}+\dfrac{\partial y_s}{\partial t}\right)}\right\} \tag{9b}$$

$$v(x,y^*,t)\Big|_{y^*=y_s^*}=\frac{2}{(\gamma+1)}\left\{\frac{\partial y_s}{\partial x}\frac{\partial y_b}{\partial x}+1\right\}\left\{\frac{\dfrac{\partial y_s}{\partial x}+\dfrac{\partial y_s}{\partial t}}{\left(\dfrac{\partial y_s}{\partial x}\right)^2+1}-\frac{1}{M_\infty^2\left(\dfrac{\partial y_s}{\partial x}+\dfrac{\partial y_s}{\partial t}\right)}\right\}-\frac{\partial y_b}{\partial x}-\frac{\partial y_b}{\partial t} \tag{9c}$$

$$p(x,y^*,t)\Big|_{y^*=y_s^*}=\left\{\frac{1}{\gamma}-\frac{2}{\gamma+1}\right\}\frac{1}{M_\infty^2}+\frac{2}{\gamma+1}\frac{\left\{\dfrac{\partial y_s}{\partial x}+\dfrac{\partial y_s}{\partial t}\right\}^2}{\left\{\left(\dfrac{\partial y_s}{\partial x}\right)^2+1\right\}} \tag{9d}$$

To complete the formulation, the condition of shock attachment, Eqs. (2) and (3), must be included. This becomes

$$y_s^*(0,t)=\left(\frac{\partial y_s^*}{\partial t}\right)_{x=0}=0 \tag{10}$$

III. Perturbation Procedure

For small harmonic oscillations of the airfoil, the upper surface may be described approximately by

$$y=y_b(x,t)=y_{b0}(x)-\alpha e^{ikt}f_b(x) \tag{11}$$

where $y=y_{b0}(x)$ defines the mean airfoil in the steady flow; α is a measure of the departure of the oscillatory flow from the mean steady flow, assumed small compared to unity; $k=\omega\bar{L}/\bar{U}_\infty$ is the reduced frequency (ω: circular frequency of the oscillation); and f_b is an arbitrary, but prescribed, function representing the mode shape of the oscillation. Here and throughout this investigation all physical quantities are given by the real part of their complex representations. The special case of a rigid airfoil oscillating with reduced frequency k about a pivot on the x axis located at a distance h downstream of the leading edge can be recovered from Eq. (11) by putting $f_b(x)=x-h$.

The resulting shock wave is assumed to oscillate with small amplitude and the same frequency as the airfoil. Then the equation defining the instantaneous position of the shock is approximately

$$y=y_s(x,t)=y_{s0}(x)-\alpha e^{ikt}f_s(x) \tag{12}$$

where $y=y_{s0}(x)$ is the shock shape when the airfoil is in its mean position and f_s is an amplitude function. In general, the functions y_{s0} and f_s are unknowns to be determined as part of the solution.

Assuming that the oscillations cause only small disturbances to the mean steady flow, and in view of the form of Eqs. (11) and (12), solutions of the boundary value problem, Eqs. (7-10) are sought in the form

$$\rho(x,y^*,t)=\rho_0(x,y^*)+\alpha e^{ikt}\rho_1(x,y^*)+O(\alpha^2) \tag{13}$$

with similar expansions for u,v^*, and p. The subscripts 0 and 1 refer to the mean steady flow and the first-order unsteady perturbed flow, respectively.

A. First-Order Flow Equations and Boundary Conditions

Substituting Eqs. (11-13) into Eq. (7) and equating like powers of α yields two sets of equations. The first, corresponding to terms independent of α, are the usual steady flow equations. The second set, corresponding to the first-order perturbed flow, may be written as:

$$\left\{u_0\frac{\partial}{\partial x}+v_0^*\frac{\partial}{\partial y^*}\right\}\rho_1+\rho_0\frac{\partial u_1}{\partial x}+\rho_0\frac{\partial v_1^*}{\partial y^*}+\left\{ik+\frac{\partial u_0}{\partial x}+\frac{\partial v_0^*}{\partial y^*}\right\}\rho_1+\frac{\partial \rho_0}{\partial x}u_1+\frac{\partial \rho_0}{\partial y^*}v_1^*=0 \tag{14a}$$

$$\left\{u_0\frac{\partial}{\partial x}+v_0^*\frac{\partial}{\partial y^*}\right\}u_1+\frac{1}{\rho_0}\frac{\partial p_1}{\partial x}-\frac{y'_{b0}}{\rho_0}\frac{\partial p_1}{\partial y^*}-\frac{1}{\rho_0^2}\left\{\frac{\partial p_0}{\partial x}-y'_{b0}\frac{\partial p_0}{\partial y^*}\right\}\rho_1+\left\{ik+\frac{\partial u_0}{\partial x}\right\}u_1+\frac{\partial u_0}{\partial y^*}v_1^*=-\frac{1}{\rho_0}\frac{\partial p_0}{\partial y^*}f'_b(x) \tag{14b}$$

$$\left\{u_0\frac{\partial}{\partial x}+v_0^*\frac{\partial}{\partial y^*}\right\}v_1^*-\frac{y'_{b0}}{\rho_0}\frac{\partial p_1}{\partial x}+\frac{1}{\rho_0}\left\{y'^2_{b0}+1\right\}\frac{\partial p_1}{\partial y^*}+\frac{1}{\rho_0^2}\left\{y'_{b0}\frac{\partial p_0}{\partial x}-(y'^2_{b0}+1)\frac{\partial p_0}{\partial y^*}\right\}\rho_1+\left\{2u_0y''_{b0}+\frac{\partial v_0^*}{\partial x}\right\}u_1$$

$$+\left\{ik+\frac{\partial v_0^*}{\partial y^*}\right\}v_1^*=u_0^2f''_b(x)+\left\{u_0\frac{\partial u_0}{\partial x}+v_0^*\frac{\partial u_0}{\partial y^*}+2iku_0+\frac{y'_{b0}}{\rho_0}\frac{\partial p_0}{\partial y^*}\right\}f'_b(x)-k^2f_b(x) \tag{14c}$$

$$\left\{u_0\frac{\partial}{\partial x}+v_0^*\frac{\partial}{\partial y^*}\right\}p_1+a_0^2\rho_0\frac{\partial u_1}{\partial x}+a_0^2\rho_0\frac{\partial v_1^*}{\partial y^*}+a_0^2\left\{\frac{\partial u_0}{\partial x}+\frac{\partial v_0^*}{\partial y^*}-\frac{u_0}{p_0}\frac{\partial p_0}{\partial x}-\frac{v_0^*}{p_0}\frac{\partial p_0}{\partial y^*}\right.$$

$$\left.+\frac{u_0}{\rho_0}(\gamma+1)\frac{\partial\rho_0}{\partial x}+\frac{v_0^*}{\rho_0}(\gamma+1)\frac{\partial\rho_0}{\partial y^*}\right\}\rho_1+\frac{\partial p_0}{\partial x}u_1+\frac{\partial p_0}{\partial y^*}v_1^*-\left\{\frac{\gamma}{\rho_0}\left(u_0\frac{\partial}{\partial x}+v_0^*\frac{\partial}{\partial y^*}\right)\rho_0-ik\right\}p_1=0 \tag{14d}$$

where primes denote differentiation with respect to x and $a_0^2=\gamma p_0/\rho_0$ is the speed of sound in the mean steady flow.

The boundary conditon (8) at the airfoil surface yields

$$v_1^*(x,0)=0 \tag{15}$$

Equations (11) and (12) imply that

$$y_s^*(x,t)=y_{s0}(x)-y_{b0}(x)-\alpha e^{ikt}\{f_s(x)-f_b(x)\}+O(\alpha^2)$$

Expanding the left-hand sides of Eq. (9) in a Taylor series about $y^*=y_{s0}(x)-y_{b0}(x)$ and employing Eqs. (8*) and (11-13) yields the shock conditions for the perturbed flow

$$\rho_1(x,y_{s0}-y_{b0})=\{f_s(x)-f_b(x)\}\frac{\partial\rho_0}{\partial y^*}(x,y_{s0}-y_{b0})-\frac{\dfrac{4(\gamma+1)y'_{s0}}{(\gamma-1)^2M_\infty^2}\{f'_s(x)+ik(y'^2_{s0}+1)f_s(x)\}}{\left\{\dfrac{2}{(\gamma-1)M_\infty^2}(y'^2_{s0}+1)+y'^2_{s0}\right\}^2} \tag{16a}$$

$$u_1(x,y_{s0}-y_{b0})=\{f_s(x)-f_b(x)\}\frac{\partial u_0}{\partial y^*}(x,y_{s0}-y_{b0})+\frac{2}{\gamma+1}\left\{\frac{2y'_{s0}f'_s(x)}{(y'^2_{s0}+1)^2}+\frac{iky'_{s0}f_s(x)}{y'^2_{s0}+1}+\frac{ikf_s(x)}{M_\infty^2y'_{s0}}\right\} \tag{16b}$$

$$v_1^*(x,y_{s0}-y_{b0})=\{f_s(x)-f_b(x)\}\frac{\partial v_0^*}{\partial y^*}(x,y_{s0}-y_{b0})+ikf_b(x)+f'_b(x)+\frac{2}{\gamma+1}\left\{\frac{2y'_{s0}f'_s(x)}{(y'^2_{s0}+1)^2}[y'_{s0}(x)-y'_{b0}(x)]\right.$$

$$\left.-\frac{f'_s(x)+y'^2_{s0}f'_b(x)}{y'^2_{s0}+1}-\frac{f'_s(x)-y'^2_{s0}f'_b(x)}{M_\infty^2y'^2_{s0}}-ikf_s(x)[1+y'_{s0}y'_{b0}]\left[\frac{1}{y'^2_{s0}+1}+\frac{1}{M_\infty^2y'^2_{s0}}\right]\right\} \tag{16c}$$

$$p_1(x,y_{s0}-y_{b0})=\{f_s(x)-f_b(x)\}\frac{\partial p_0}{\partial y^*}(x,y_{s0}-y_{b0})-\frac{4y'_{s0}}{(\gamma+1)(y'^2_{s0}+1)^2}\{f'_s(x)+ik(y'^2_{s0}+1)f_s(x)\} \tag{16d}$$

We note that to the order of magnitude retained, the shock conditions are applied at the mean shock location. Finally, the condition of shock attachment (10) becomes

$$f_s(0)=f_b(0) \tag{17}$$

B. Von Mises Transformation

The steady flow equation of continuity implies the existence of a stream function ψ_0 defined by

$$\frac{\partial\psi_0}{\partial x}=-\rho_0v_0^* \qquad \frac{\partial\psi_0}{\partial y^*}=\rho_0u_0 \tag{18}$$

Applying the Von Mises transformation from (x,y^*) to (x,ψ_0) via Eq. (18), followed by a second transformation given by

$$x=x, \qquad \psi=\psi_0-y_{s0}(x) \tag{19}$$

allows Eq. (14) to be written more conveniently in the matrix form

$$M_1\frac{\partial W}{\partial x}+M_2\frac{\partial W}{\partial\psi}+M_3W=M_4 \tag{20}$$

where

$$W=\begin{bmatrix}\rho_1\\ u_1\\ v_1^*\\ p_1\end{bmatrix}\quad M_1=\begin{bmatrix}u_0 & \rho_0 & 0 & 0\\ 0 & u_0 & 0 & 1/\rho_0\\ 0 & 0 & u_0 & -y'_{b0}/\rho_0\\ 0 & a_0^2\rho_0 & 0 & u_0\end{bmatrix}\quad M_2=\begin{bmatrix}-u_0y'_{s0} & -\rho_0[y'_{s0}+\rho_0v_0^*] & \rho_0^2u_0 & 0\\ 0 & -u_0y'_{s0} & 0 & -[y'_{b0}u_0+v_0^*+y'_{s0}/\rho_0]\\ 0 & 0 & -u_0y'_{s0} & \left[(y'^2_{b0}+1)u_0+y'_{b0}v_0^*+\dfrac{y'_{s0}y'_{b0}}{\rho_0}\right]\\ 0 & -a_0^2\rho_0[y'_{s0}+\rho_0v_0^*] & a_0^2\rho_0^2u_0 & -u_0y'_{s0}\end{bmatrix}$$

$$M_3=\begin{bmatrix}ik+a_1 & a_2 & a_3 & 0\\ a_4 & ik+a_5 & a_6 & 0\\ a_7 & a_8 & ik+a_9 & 0\\ 0 & a_{11} & a_{12} & ik+a_{13}\end{bmatrix},\quad M_4=\begin{bmatrix}0\\ -u_0\dfrac{\partial p_0}{\partial\psi}f'_b(x)\\ u_0^2f''_b(x)+[2iku_0+a_{10}]f'_b(x)-k^2f_b(x)\\ 0\end{bmatrix} \tag{21}$$

wherein the functions a_i $(i=1,2,...,13)$ depend solely and explicitly on the steady-state functions $u_0,\rho_0,v_0^*,p_0,y_{s0},y_{b0}$, and their derivatives (see the Appendix).

In Eqs. (20) and (21) the functions y_{b0} and f_b are arbitrary. Hence, this equation is valid for all airfoil shapes and for both general rigid and nonrigid motions of the airfoil. These equations exhibit a good deal more generality than those by Hui,[7] Carrier,[8] and Van Dyke[9] in their treatments of the oscillating wedge problems.

IV. Rigid and Nonrigid Wedge Motions

The foregoing equations (20) are complicated because they apply to the case of arbitrary airfoil shape and arbitrary mode of oscillation. To illustrate a method of solution, we consider the relatively simpler case of an oscillating rigid or deformable wedge (not necessarily thin).

Let the wedge surface in the steady flow be defined by

$$y=y_{b0}(x)\equiv ax$$

Then, the slope b of the wedge-shaped shock may be computed from the relation

$$a=\frac{2[1+ab]}{b(\gamma+1)}\left[\frac{b^2}{b^2+1}-\frac{1}{M_\infty^2}\right]$$

For the wedge, all steady flow quantities are constant. Hence, all a_i's $(1\le i\le 13)$ are identically zero in Eqs. (21).

A. The Canonical System for the Perturbed Flow

Defining

$$\hat u_1=\rho_0u_1\qquad \hat v_1^*=\rho_0v_1^* \tag{22}$$

Eq. (20) becomes

$$A_1\frac{\partial\hat W}{\partial x}+A_2\frac{\partial\hat W}{\partial\psi}+ik\hat W=C \tag{23}$$

where

$$\hat W=\begin{bmatrix}\rho_1\\ \hat u_1\\ \hat v_1^*\\ p_1\end{bmatrix}\quad A_1=\begin{bmatrix}u_0 & 1 & 0 & 0\\ 0 & u_0 & 0 & 1\\ 0 & 0 & u_0 & -a\\ 0 & a_0^2 & 0 & u_0\end{bmatrix}$$

$$A_2=\begin{bmatrix}-bu_0 & -b & \rho_0u_0 & 0\\ 0 & -bu_0 & 0 & -b-a\rho_0u_0\\ 0 & 0 & -bu_0 & ab+[a^2+1]\rho_0u_0\\ 0 & -ba_0^2 & a_0^2\rho_0u_0 & -bu_0\end{bmatrix}$$

$$C=\begin{bmatrix}0\\ 0\\ \rho_0u_0^2f''_b(x)+2ik\rho_0u_0f'_b(x)-k^2\rho_0f_b(x)\\ 0\end{bmatrix} \tag{24}$$

From Eqs. (24), det $A_2\neq 0$ provided

$$b\neq a_0^2\rho_0u_0[a\pm\beta]/(u_0^2-a_0^2) \tag{25}$$

where $\beta^2=M_0^2-1$. Hence, Eq. (23) may be written as:

$$\frac{\partial\hat W}{\partial\psi}+A\frac{\partial\hat W}{\partial x}+ikA_2^{-1}\hat W=A_2^{-1}C \tag{26}$$

where $A\equiv A_2^{-1}A_1$. Matrix A has eigenvalues

$$\lambda_1=\lambda_2=-1/b$$

$$\lambda_{3,4}=\frac{b^2u_0^2}{\det A_2}\{(a_0^2-u_0^2)b+aa_0^2\rho_0u_0\pm a_0^2\rho_0u_0\beta\} \tag{27}$$

The corresponding eigenvectors of A yield a nonsingular matrix

$$P=\begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 1 & -H^- & -H^+\\ 0 & 0 & -\beta a_0^2/u_0 & \beta a_0^2/u_0\\ 0 & 0 & a_0^2 & a_0^2\end{bmatrix}$$

where

$$H^\pm=\frac{a_0^2\rho_0+abu_0\pm a_0^2(b+a\rho_0u_0)\beta/u_0}{ab+(a^2+1)\rho_0u_0\pm b\beta} \tag{28}$$

such that

$$P^{-1}AP=\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$$

Defining

$$Z=(Z_i)\equiv P^{-1}\hat W$$

$$=\begin{bmatrix}\rho_1-p_1/a_0^2\\ \hat u_1-u_0[H^--H^+]\hat v_1^*/2a_0^2\beta+[H^-+H^+]p_1/2a_0^2\\ -u_0\hat v_1^*/2a_0^2\beta+p_1/2a_0^2\\ u_0\hat v_1^*/2a_0^2\beta+p_1/2a_0^2\end{bmatrix} \tag{29}$$

and multiplying Eq. (26) on the left by P^{-1} yields the canonical system

$$\frac{\partial Z}{\partial \psi} + \Lambda \frac{\partial Z}{\partial x} + ikRZ = TC \tag{30}$$

where

$$\Lambda = P^{-1}AP = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

$$T = P^{-1}A_2^{-1} \equiv (t_{ij})$$

$$R = P^{-1}A_2^{-1}P = TP \equiv (r_{ij})$$

Again, expressions for t_{ij} and r_{ij} can be found in the Appendix.

B. Boundary Conditions, Method of Solution

The basic equation to be solved is Eq. (30). In most engineering problems, the reduced frequency parameter k is small, of order of magnitude 0.01. Hence, we seek a solution of Eq. (30) in the form

$$Z(x,\psi) = Z^{(0)}(x,\psi) + ikZ^{(1)}(x,\psi) + (ik)^2 Z^{(2)}(x,\psi) + \ldots \tag{31}$$

Substituting Eq. (31) in Eq. (30) and equating like powers of ik yields, for all $n \geq 0$ and for $\nu = 1, 2, 3, 4$,

$$\frac{\partial Z_\nu^{(n)}}{\partial \psi} + \lambda_\nu \frac{\partial Z_\nu^{(n)}}{\partial x} + D_\nu^{(n)} = 0 \tag{32a}$$

where

$$D^{(n)}(x,\psi) = RZ^{(n-1)} - TC^{(n)} \tag{32b}$$

with

$$C^{(0)} = \begin{bmatrix} 0 \\ 0 \\ \rho_0 u_0^2 f_b''(x) \\ 0 \end{bmatrix} \quad C^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 2\rho_0 u_0 f_b'(x) \\ 0 \end{bmatrix}$$

$$C^{(2)} = \begin{bmatrix} 0 \\ 0 \\ \rho_0 f_b(x) \\ 0 \end{bmatrix} \quad C^{(p)} \equiv 0, \quad \text{for all } p \geq 3 \tag{32c}$$

and $Z^{(-1)} \equiv$ zero matrix.

Equations (16) in conjunction with Eq. (31) yield the following conditions to be applied at the mean shock position, now given by $\psi = 0$:

$$Z_2^{(0)}(x,0) = B_1 f_b'(x) + B_2 f_s'^{(0)}(x)$$

$$Z_3^{(0)}(x,0) = C_1 f_b'(x) + C_2^+ f_s'^{(0)}(x)$$

$$Z_4^{(0)}(x,0) = -C_1 f_b'(x) - C_2^- f_s'^{(0)}(x) \tag{33a}$$

and

$$Z_3^{(1)}(x,0) = C_3 f_b(x) + C_4^- f_s^{(0)}(x) + C_2^+ f_s'^{(1)}(x)$$

$$Z_4^{(1)}(x,0) = -C_3 f_b(x) - C_4^+ f_s^{(0)}(x) - C_2^- f_s'^{(1)}(x) \tag{33b}$$

Similar expressions may be obtained for $Z_1^{(0)}$, $Z_1^{(1)}$, and $Z_2^{(1)}$. However, our interest lies in determining the pressure distribution which, in view of Eq. (29), is given in terms of Z_3 and Z_4. Due to the form of the matrix R (see Eqs. (30), and (32b), and the Appendix), $Z_2^{(0)}$ is required for the computation of $Z_3^{(1)}$ and $Z_4^{(1)}$.

To obtain Eqs. (33), we have written

$$f_s(x) = f_s^{(0)}(x) + ikf_s^{(1)}(x) + O(k^2) \tag{34}$$

The constant coefficients in Eqs. (33) are given by:

$$B_1 = \frac{\rho_0 u_0}{(\gamma+1)a_0^2\beta}(H^+ - H^-)\left[\frac{1}{M_\infty^2} - \frac{b^2}{b^2+1} + \frac{\gamma+1}{2}\right]$$

$$B_2 = \frac{1}{\gamma+1}\left[\frac{4b\rho_0}{(b^2+1)^2} - \frac{2b(H^+ + H^-)}{a_0^2(b^2+1)^2} + \frac{\rho_0 u_0}{a_0^2\beta}(H^- - H^+)\left\{\frac{1}{M_\infty^2 b^2} + \frac{1}{b^2+1} - \frac{2b(b-a)}{(b^2+1)^2}\right\}\right]$$

$$C_1 = \frac{-B_1}{H^+ - H^-}$$

$$C_2^\pm = -\frac{1}{a_0^2(\gamma+1)}\left[\frac{u_0\rho_0}{\beta}\left\{\frac{2b(b-a)}{(b^2+1)^2} - \frac{1}{b^2+1} - \frac{1}{M_\infty^2 b^2}\right\} \pm \frac{2b}{(b^2+1)^2}\right]$$

$$C_3 = -\frac{u_0\rho_0}{2a_0^2\beta}$$

$$C_4^\pm = \frac{1}{a_0^2(\gamma+1)}\left[\frac{u_0\rho_0(1+ab)}{\beta}\left\{\frac{1}{M_\infty^2 b^2} + \frac{1}{b^2+1}\right\} \pm \frac{2b}{b^2+1}\right] \tag{35}$$

To obtain the boundary conditions at the wedge surface, we first note that Eq. (29) implies

$$\hat{v}_1^* = (a_0^2\beta/u_0)(Z_4 - Z_3)$$

Hence Eq. (15), together with Eqs. (19), (22), and (31), gives the condition

$$Z_4^{(n)} - Z_3^{(n)} = 0 \text{ at } \psi = -bx \tag{36}$$

for all $n \geq 0$.

The Cauchy problem presented by Eqs. (32) and (33) can be solved by the following method.[11] Initial values for $Z_\nu^{(n)}$ are prescribed along the positive x axis, (i.e., $\psi = 0$) by Eq. (33). The equations of the characteristics through the point (ξ, η) in the (x, ψ) plane are

$$x = X_k(\psi; \xi, \eta) \equiv \xi - \lambda_k(\eta - \psi) \quad k = 1,2,3,4 \tag{37}$$

Using ψ as a parameter along the characteristics, Eqs. (32) can be integrated to yield

$$Z_\nu^{(n)}(\xi,\eta) = Z_\nu^{(n)}(X_\nu(0;\xi,\eta),0) - \int_0^\eta D_\nu^{(n)}(X_\nu(\sigma;\xi,\eta),\sigma)\,\mathrm{d}\sigma \tag{38}$$

for all $n \geq 0$ and $\nu = 1, 2, 3, 4$.

We note from Eq. (32b) that $D^{(n)}$ depends on $Z^{(n-1)}$. Hence, Eq. (38) shows that the nth approximation $Z^{(n)}$ can be found by performing an integration involving the previous approximation $Z^{(n-1)}$. Also, Eq. (38) incorporates boundary conditions (33) but not (36). In fact, the present problem has an additional unknown function, f_s, determined using condition Eq. (36) at the wedge surface.

C. Solution for Pressure Field

From Eq. (29) the pressure is determined as

$$p_1^{(n)}/a_0^2 = Z_3^{(n)} + Z_4^{(n)}$$

where we have expanded p_1 in powers of ik:

$$p_1 = p_1^{(0)} + ikp_1^{(1)} + \ldots$$

1) $n=0$: In this case, Eq. (32b) gives

$$D^{(0)} = -TC^{(0)} = -\rho_0 u_0^2 f_b''(x) \begin{bmatrix} 0 \\ t_{23} \\ t_{33} \\ t_{43} \end{bmatrix}$$

and hence Eq. (38) gives, for $\nu = 1, 2, 3, 4$,

$$Z_\nu^{(0)}(\xi,\eta) = Z_\nu^{(0)}(X_\nu(0;\xi,\eta),0) + \rho_0 u_0^2 \begin{bmatrix} 0 \\ t_{23} \\ t_{33} \\ t_{43} \end{bmatrix} \int_0^\eta f_b''(X_\nu(\sigma;\xi,\eta))\,d\sigma$$

For our purposes, we need only find $Z_\nu^{(0)}$ for $\nu = 2, 3, 4$. Using Eqs. (37) and (33), we obtain

$$Z_2^{(0)}(\xi,\eta) = B_2 f_s'^{(0)}(\xi+\eta/b) - t_{23} b\rho_0 u_0^2 f_b'(\xi) + [B_1 + t_{23} b\rho_0 u_0^2] f_b'(\xi+\eta/b) \tag{39a}$$

$$Z_3^{(0)}(\xi,\eta) = C_2^+ f_s'^{(0)}(\xi-\lambda_3\eta) - \frac{\rho_0 u_0^2}{2a_0^2\beta} f_b'(\xi) + \left[C_1 + \frac{\rho_0 u_0^2}{2a_0^2\beta}\right] f_b'(\xi-\lambda_3\eta) \tag{39b}$$

$$Z_4^{(0)}(\xi,\eta) = -C_2^- f_s'^{(0)}(\xi-\lambda_4\eta) + \frac{\rho_0 u_0^2}{2a_0^2\beta} f_b'(\xi) - \left[C_1 + \frac{\rho_0 u_0^2}{2a_0^2\beta}\right] f_b'(\xi-\lambda_4\eta) \tag{39c}$$

Equations (39) contain the unknown functions $f_s^{(0)}$ which can be determined by applying the boundary condition (36). In doing so, one arrives at the functional equation

$$f_s'^{(0)}\left(\frac{\phi}{\Gamma}\right) + \bar{\lambda} f_s'^{(0)}(\phi) = E_1 f_b'\left(\frac{\phi}{1+\lambda_3 b}\right) + E_2\left[f_b'\left(\frac{\phi}{\Gamma}\right) + f_b'(\phi)\right] \tag{40}$$

where

$$\phi = (1+\lambda_3 b)\xi, \quad \Gamma = \frac{1+\lambda_3 b}{1+\lambda_4 b}, \quad \bar{\lambda} = C_2^+/C_1^-, \quad E_1 = \rho_0 u_0^2/(a_0^2\beta C_2^-), \quad E_2 = -\left[C_1 + \frac{\rho_0 u_0^2}{2a_0^2\beta}\right]\Big/ C_2^- \tag{41}$$

Physically, the quantities $\bar{\lambda}$ and Γ represent an attenuation factor and the reflection coefficient for the waves undergoing multiple reflections from the bow shock wave.[4] It can be easily shown that $|\bar{\lambda}| < 1$ and $|\Gamma| < 1$.

The solution of Eq. (40) satisfying the attachment condition (17) is:

$$f_s^{(0)}(x) = f_b(0) + E_2\{f_b(x) - f_b(0)\} + E_1(1+\lambda_4 b)\sum_{n=0}^{\infty}(-1)^n(\bar{\lambda}/\Gamma)^n\left\{f_b\left(\frac{\Gamma^{n+1}x}{1+\lambda_3 b}\right) - f_b(0)\right\}$$
$$+ E_2\frac{(1-\bar{\lambda})}{\Gamma}\sum_{n=0}^{\infty}(-1)^n(\bar{\lambda}/\Gamma)^n\{f_b(\Gamma^{n+1}x) - f_b(0)\} \tag{42}$$

Equation (42) gives the zeroth-order shock shape and is actually the integral of the solution of Eq. (40). Using Eq. (42) in Eqs. (39) gives expressions for $Z_\nu^{(0)}$ and hence $p_1^{(0)}$ in terms of the mode shape $f_b(x)$ of the wedge oscillations.

2) $n=1$: Since our main objective is to determine $p_1^{(1)}$, we are concerned here only with $Z_3^{(1)}$ and $Z_4^{(1)}$. From Eqs. (32b and c), for $\nu = 3, 4$, one obtains

$$D_\nu^{(1)} = \sum_{i=2}^{4} r_{\nu i} Z_i^{(0)} - 2t_{\nu 3}\rho_0 u_0 f_b'$$

Proceeding as in the case $n=0$ yields a functional equation of the form

$$f_s'^{(1)}\left(\frac{\phi}{\Gamma}\right) + \bar{\lambda} f_s'^{(1)}(\phi) = K_1^- f_b(0) + K_2^- f_b\left(\frac{\phi}{1+\lambda_3 b}\right) + G_1(\phi) + G_2\left(\frac{\phi}{\Gamma}\right) \tag{43a}$$

where

$$G_1(\phi) = K_3^- f_b(\phi) + K_5^- f_s^{(0)}(\phi) + \frac{K_7^+}{(1+\lambda_3 b)}\phi f_b'(\phi) + \frac{K_9^+}{(1+\lambda_3 b)}\phi f_s'^{(0)}(\phi)$$

$$G_2(\phi) = K_4^- f_b(\phi) + K_6^- f_s^{(0)}(\phi) + \frac{K_8^+}{(1+\lambda_4 b)}\phi f_b'(\phi) + \frac{K_{10}^+}{(1+\lambda_4 b)}\phi f_s'^{(0)}(\phi) \tag{43b}$$

and

$$C_2^- K_1^\pm = \left\{ \frac{bt_{32}}{(1+\lambda_3 b)} \pm \frac{bt_{42}}{(1+\lambda_4 b)} \right\} \left\{ B_1 + B_2 + b\rho_0 u_0^2 t_{23} \right\}$$

$$C_2^- K_2^\pm = -b\rho_0 u_0^2 t_{23} \left\{ \frac{t_{32}}{\lambda_3} \pm \frac{t_{42}}{\lambda_4} \right\} - 2\rho_0 u_0 \left\{ \frac{t_{33}}{\lambda_3} \pm \frac{t_{43}}{\lambda_4} \right\} + \frac{\rho_0 u_0^2}{2a_0^2 \beta \lambda_3} \{ r_{34} - r_{33} \} \pm \frac{\rho_0 u_0^2}{2a_0^2 \beta \lambda_4} \{ r_{44} - r_{43} \}$$

$$C_2^- K_3^\pm = -C_3 - \frac{bB_1 t_{32}}{1+\lambda_3 b} + \frac{b\rho_0 u_0^2 t_{32} t_{23}}{\lambda_3 (1+\lambda_3 b)} + \frac{2\rho_0 u_0 t_{33}}{\lambda_3} + \frac{\rho_0 u_0^2}{2a_0^2 \beta \lambda_3} \{ r_{33} - r_{34} \} - \frac{1}{(\lambda_4 - \lambda_3)} \left\{ C_1 + \frac{\rho_0 u_0^2}{2a_0^2 \beta} \right\} \left\{ r_{34} \mp r_{43} \right\}$$

$$C_2^- K_4^\pm = \pm C_3 \mp \frac{bB_1 t_{42}}{1+\lambda_4 b} \pm \frac{b\rho_0 u_0^2 t_{42} t_{23}}{\lambda_4 (1+\lambda_4 b)} \pm \frac{2\rho_0 u_0 t_{43}}{\lambda_4} \mp \frac{\rho_0 u_0^2}{2a_0^2 \beta \lambda_4} \{ r_{44} - r_{43} \} - \frac{1}{(\lambda_4 - \lambda_3)} \left\{ C_1 + \frac{\rho_0 u_0^2}{2a_0^2 \beta} \right\} \left\{ r_{34} \pm r_{43} \right\}$$

$$C_2^- K_5^\pm = -C_4^- - \frac{bB_2 t_{32}}{1+\lambda_3 b} - \frac{1}{(\lambda_4 - \lambda_3)} \{ C_2^- r_{34} \mp C_2^+ r_{43} \}, \quad C_2^- K_6^\pm = \pm C_4^+ \mp \frac{bB_2 t_{42}}{1+\lambda_4 b} + \frac{1}{(\lambda_4 - \lambda_3)} \{ C_2^- r_{34} \mp C_2^+ r_{43} \}$$

$$C_2^- K_7^\pm = -br_{33} \left\{ C_1 + \frac{\rho_0 u_0^2}{2a_0^2 \beta} \right\}, \quad C_2^- K_8^\pm = -br_{44} \left\{ C_1 + \frac{\rho_0 u_0^2}{2a_0^2 \beta} \right\}, \quad K_9^\pm = -\bar{\lambda} br_{33}, \quad K_{10}^\pm = -br_{44} \tag{43c}$$

The solution of Eqs. (43) is:

$$f_s'^{(1)}(x) = \frac{K_1^-}{1+\bar{\lambda}} f_b(0) + K_2^- \sum_{n=0}^{\infty} (-1)^n \bar{\lambda}^n f_b \left(\frac{\Gamma^{n+1} x}{1+\lambda_3 b} \right) + \sum_{n=0}^{\infty} (-1)^n \bar{\lambda}^n G_1(\Gamma^{n+1} x) + \sum_{n=0}^{\infty} (-1)^n \bar{\lambda}^n G_2(\Gamma^n x) \tag{44}$$

Calculating $Z_3^{(1)}$ and $Z_4^{(1)}$ from Eq. (38), the pressue (to first order) along the wedge surface $\eta = -b\xi$ is:

$$\frac{1}{a_0^2} p_1^{(1)} \Big|_{\text{body}} = Z_3^{(1)}(\xi, -b\xi) + Z_4^{(1)}(\xi, -b\xi) = C_2^+ f_s'^{(1)}([1+\lambda_3 b]\xi) - C_2^- f_s'^{(1)}([1+\lambda_4 b]\xi) + L_1 f_b(0) + L_2 f_b(\xi)$$
$$+ L_3 f_b([1+\lambda_3 b]\xi) + L_4 f_b([1+\lambda_4 b]\xi) + L_5 f_s^{(0)}([1+\lambda_3 b]\xi) + L_6 f_s^{(0)}([1+\lambda_4 b]\xi) + L_7 \xi f_b'([1+\lambda_3 b]\xi)$$
$$+ L_8 \xi f_b'([1+\lambda_4 b]\xi) + L_9 \xi f_s'^{(0)}([1+\lambda_3 b]\xi) - L_{10} \xi f_s'^{(0)}([1+\lambda_4 b]\xi) \tag{45}$$

where $L_i = -C_2^- K_i^+$, and $f_s'^{(1)}$, $f_s^{(0)}$ are given by Eqs. (44) and (42), respectively.

We emphasize that f_b may equally well represent pitching or plunging of the wedge as a whole, a local perturbation on the wedge surface, or elastic deformations of the wedge. The pressure distribution, Eq. (45), is valid for wedges of any thickness (provided the shock wave is attached) and reduces to the results of McIntosh[4] in the thin wedge approximation. In the case of pitching ($f_b(x) = x - h$), Eq. (45) corresponds to the solution given by Hui.[7] In the hypersonic limit, Eq. (45) reduces to the results of Hui.[6] Elastic deformations may be described by appropriately choosing $f_b(x)$; hence, Eq. (45) should serve as a useful guide to the aeroelastician.

V. Concluding Remarks

A perturbation procedure is used to obtain a general matrix formulation describing first-order effects due to oscillating (curved) airfoils in a supersonic or hypersonic stream. The basic theory is valid for both sharp and blunt bodies and thus encompasses and enlarges on the theories of Hui,[6,7] Carrier,[8] Van Dyke,[9] and others; all their results being special cases of the present. Unsteady wedge motions are considered, e.g., pitching, elastic deformations, etc., and the equations are integrated to yield the pressure distribution on the wedge surface.

The method of analysis which is presented readily lends itself to the development of an unsteady Newtonian theory suitable for treating the problem of oscillating blunt-nosed airfoils in a hypersonic flow. Such a problem is considered in part II of this paper.[10]

Appendix

The quantities a_i $(i = 1,\ldots,13)$ in M_3 and M_4 are:

$$a_1 = -(u_0/\rho_0) D\rho_0, \quad a_2 = D\rho_0 - \rho_0 v_0^* \partial\rho_0/\partial\psi$$

$$a_3 = \rho_0 u_0 \partial p_0/\partial\psi, \quad a_4 = (1/\rho_0) Du_0$$

$$a_5 = Du_0 - \rho_0 v_0^* \partial u_0/\partial\psi, \quad a_6 = \rho_0 u_0 \partial u_0/\partial\psi$$

$$a_7 = -(u_0 y_{b0}'/\rho_0) Du_0 - (u_0/\rho_0) \partial p_0/\partial\psi$$

$$a_8 = 2u_0 y_{b0}'' + Dv_0^* - \rho_0 v_0^* \partial v_0^*/\partial\psi, \quad a_9 = \rho_0 u_0 \partial v_0^*/\partial\psi$$

$$a_{10} = u_0 Du_0 - u_0 y_{b0}' \partial p_0/\partial\psi, \quad a_{11} = Dp_0 - \rho_0 v_0^* \partial p_0/\partial\psi$$

$$a_{12} = \rho_0 u_0 \partial p_0/\partial\psi, \quad a_{13} = -(\gamma u_0/\rho_0) D\rho_0$$

where

$$D \equiv \frac{\partial}{\partial x} - y_{s0}' \frac{\partial}{\partial\psi}$$

The matrices T and R have elements (only relevant ones are listed):

$$t_{11} = -a_0^2 t_{14} = -1/(bu_0)$$

$$t_{12} = t_{13} = t_{21} = t_{31} = t_{41} = r_{12} = r_{13} = r_{14} = r_{21} = r_{31} = r_{41} = 0$$

$$t_{23} = \frac{bu_0^2}{\det A_2} \left\{ \rho_0 a_0^2 (b + a\rho_0 u_0) + \frac{b}{2a_0^2 \beta} [H^- - H^+] \right.$$
$$\left. \times [(u_0^2 - a_0^2) b - a a_0^2 \rho_0 u_0] - \frac{bu_0 \rho_0}{2} [H^- + H^+] \right\}$$

$$t_{32} = r_{32} = -(a_0^2/u_0) t_{34}$$
$$= \frac{b^2 u_0^2}{2\beta \det A_2} \{ b\beta - ab - (a^2+1) \rho_0 u_0 \}$$

$$t_{42} = r_{42} = -(a_0^2/u_0)t_{44}$$

$$= \frac{b^2 u_0^2}{2\beta \det A_2} \{b\beta + ab + (a^2+1)\rho_0 u_0\}$$

$$t_{33} = -\lambda_3/(2a_0^2\beta)$$

$$t_{43} = \lambda_4/(2a_0^2\beta)$$

$$r_{i3} = -H^- t_{i2} - (a_0^2/u_0)\beta t_{i3} + a_0^2 t_{i4} \quad i=2,3,4$$

$$r_{14} = -H^+ t_{i2} + (a_0^2/u_0)\beta t_{i3} + a_0^2 t_{i4} \quad i=2,3,4$$

Acknowledgment

The author expresses his thanks to P. Mandl for his advice and to the referees for their useful comments.

References

[1] Lighthill, M. J., "Oscillating Airfoils at High Mach Number," *Journal of Aeronautical Sciences*, Vol. 20, June 1953, pp. 402-406.

[2] Zartarian, G., Hsu, B. T., and Ashley, H., "Dynamic Airloads and Aeroelastic Problems at Entry Mach Number," *Journal of Aeronautical Sciences*, Vol. 28, March 1961, pp. 209-222.

[3] Miles, J. W., "Unsteady Flow at Hypersonic Speeds," *Hypersonic Flow*, Butterworths Scientific Publications, London, 1960, pp. 185-198.

[4] McIntosh, Jr., S. C., "Hypersonic Flow Over an Oscillating Wedge," *AIAA Journal*, Vol. 3, March 1965, pp. 433-440.

[5] Appleton, J. P., "Aerodynamic Pitching Derivatives of a Wedge in Hypersonic Flow," *AIAA Journal*, Vol. 2, Nov. 1964, pp. 2034-2036.

[6] Hui, W. H., "General Perturbation Theory for Hypersonic and Supersonic Flows Past a Wedge," AASU Rept. No. 276, Sept. 1967.

[7] Hui, W. H., "Stability of Oscillating Wedges and Caret Wings in Hypersonic and Supersonic Flows," *AIAA Journal*, Vol. 7, Aug. 1969, pp. 1524-1530.

[8] Carrier, G. F., "The Oscillating Wedge in a Supersonic Stream," *Journal of Aeronautical Sciences*, Vol. 16, March 1949, pp. 150-152.

[9] Van Dyke, M. D., "On Supersonic Flow Past an Oscillating Wedge," *Quarterly of Applied Mathematics*, Vol. 11, No. 3, Oct. 1953, pp. 360-363.

[10] Barron, R. M. and Mandl, P., "Oscillating Airfoils: II. Newtonian Flow Theory and Application to Power-Law Bodies in Hypersonic Flow," to be published in *AIAA Journal*.

[11] Garabedian, P. R., *Partial Differential Equations*, Wiley and Sons, Inc., New York, 1964.